RESEARCH ARTICLE

On Non-Parametric Criteria for Random Communication and Processes Relationship

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ABSTRACT: The properties of partially dependent random variables, representing the extreme (turning) point of time series have been investigated. Asymptotic normality of sums of partly dependent random variables has been proven. Generalization of Kolmogorov inequality has been proven.

KEYWORDS: Extreme point, the criteria of randomness, partial dependence of the random variable

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1. INTRODUCTION

Patterns of stochastic nature prevail in the financial and economic relations as well as in any other activity [1]. For instance financial markets possess suchlike peculiarities [2]. The problem of stochastic communication measuring technique development is topical issue for a score of reasons. In particular when producing forecasts of complex processes the stochastic factor should be taken into account.

It should be noted that statistic or econometric approaches prevail in the researches. In practice majority of applied researches in the sphere of management of capital, securities management etc. the stochastic factor is introduced onto the existing models without any preliminary content-related analysis connected with investigation into identification of the objectively reasonable causes for the factor existence. Intuitive consideration precedes ordinary models reviews. In connection with the foregoing the problem of elaboration of methods and criteria which are as close as possible to the objective assessment of presence or lack of randomness factor in the interaction of financial and economic processes is relevant. Application of the methods will permit to make modeling of financial and economic processes with the help of statistic models more established. Besides application of estimation methods of randomness factor presence does not exclude but complements the necessary logical analysis.

The problem lies in the quantitative assessment of the randomness level in the impact of many factors on the successful indication.

It should be noted that development of objective assessment of presence (lack) of randomness factor in the interaction or impact of social and economic processes will permit to make the process of econometric modeling more proved. In particular the choice of modeling methods may be clarified.

The essence of the problem can be clarified with simple examples.

Example1.Interaction of socio-economic processes within the frames of economic theory can be represented in the form of particular relation having the following form:

$$\Phi(x_1, x_2, ..., x_n, y, \xi) = 0 \quad , (1)$$

in which $x_i, i = 1, 2, ..., n, y$ are variables, which characterize relevant processes (they can be either deterministic or accidental);

ξ - some random variable (with pre determined or indeterminate law of distribution);

$\Phi(..., ..., ...)$ - some function.

Randomness in the interaction of processes reflected in (1) is provided by random variable ξ.

Example2. In the theory and practice of econometric

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modelling of socio-economic processes quite often the following econometric models are used

$$y = f(x_1, x_2, ..., x_n, \xi) \text{ , (2)}$$

In which the symbols in brackets have the same meaning as in the previous formula. In this example the task is to assess the randomness level in the effect of $x_1, x_2, ..., x_n$ on y.

Difference between formulas (1) and (2) lies in the fact the first one reflects implicit relation and the second one reflects direct relation between successful variable and factor variables $x_1, x_2, ..., x_n$.

2. Synthesis of the randomness level criterion

2.1. Background information

Concerning initial statistical information, which can be used for synthesis and identification of financial-economic processes, the following assumptions have been made. Data on each factor should contain at least 9 - 12 observations. Factors measured not less than in ordinal scale are accepted. First assumption is related to chronic non sufficiency and in combination with the second one imposes restrictions on the choice of tools which can be applied to solve the problem.

2.2. Criteria for the level of randomness

The concept of extreme point has been taken as a basis of development of randomness level assessment indicator of interaction of financial-economic processes [3, p. 27].

Let the index u be set by observations $u_0, u_1, u_2, ..., u_{n-1}, n \geq 3$. We arrange observations u_i in the plane in the orthogonal coordinate system, one of the axes of which is axis of indexes *i* equidistant from each other and the second axis is the one of u indicator values changes. Thus the indexes can be interpreted as the points of the plane. It should be noted that there may be cases when such constructions are incorrect. Point $u_i, i = 1, 2, ..., n-2$ is called extreme point in case it is a peak as comparing to the two neighboring points, that is $u_{i-1} < u_i$ and $u_i > u_{i+1}$, or a trough, and thus $u_{i-1} > u_i$, and $u_i < u_{i+1}$. The concept of an extreme point is used in statistics to form the criteria of factor randomness. The level of factor randomness is characterized by the number of extreme points in the sequence of observations.

In order to formalize the fact of presence or lack of extreme point in the succession $\{u_i\}_{i=1}^{n-1}$ we will introduce the simplest random variable – counter, which takes only two values: 1 when u_i is an extreme point, 0 –in the contrary case. We shall denote the counter with the symbol x_i.

Further we shall interpret the observations $u_1, u_2, ..., u_{n-1}$ as realization of some unknown but continuous density function. It can be seen at once that $\pi_u = \sum x_i$ is a random value which takes on values 0, 1, 2, ..., $n-2$ (x_i –characteristic function for u_i). Value π_u can be accepted as a basis for the construction of a randomness level criterion and its generalizations. But first extreme points properties should be studied.

3. Extreme points properties

3.1. Some properties of the extreme point's characteristics

In [3] mathematical expectation and the value of the variance π_u have been found. They have the following form:

$$M(\pi_u) = \frac{2}{3}(n-2), \; D(\pi_u) = \frac{16n-29}{90}. \quad (3)$$

Let's consider the following method of mathematical expectation calculation and the value of the variance π_u. In the previous paragraph it has been noted that random variable x_i can be constructed using characteristic functions.

Let us U be a set of extreme points of the sequence $\{u_i\}_{i=1}^{n-1}$, and $\overline{U}$ - its complement. Then, where $x_i = I_U(u_i)$, where

$$I_U(u_i) = \begin{cases} 1, u_i \in U, \\ 0, u_i \in \overline{U} \end{cases} \text{ –Characteristic function.}$$

It can be seen that random variable x_i can be put into consideration in another way. Precisely speaking for all $i,\ 1 \le i \le n-2$ we can set $x_i = g(u_{i-1}, u_i, u_{i+1})$, where

$$g(a,b,c) = \begin{cases} 1, (b-a)(b-c) > 0, \\ 0, (b-a)(b-c) \le 0. \end{cases}$$

The fact of equivalence of methods for determining a random variable x_i follows from the understanding of the fact how set U is constructed:

$$U = \{u_i \mid (u_i - u_{i-1})(u_i - u_{i+1}) > 0, i = 1, 2, \ldots, n-2\} .$$

For $M(\pi_u)$ there are equations

$$M(\pi_u) = M(\sum_{i=1}^{n-2} x_i) = \sum_{i=1}^{n-2} M(x_i) = \sum_{i=1}^{n-2} p(x_i = 1) = \sum_{i=1}^{n-2} p(u_i \in U).$$

For extreme points u_i with probability 1 condition $(u_i - u_{i-1})(u_i - u_{i+1}) > 0$ has been fulfilled. It means that the pairs $(u_{i-1},\ u_i)$ and $(u_i,\ u_{i+1})$ are different at the same time. As far as sequences with independent random variables are dealt with density function u_i is continuous, probability of coincidence of random variables

$u_{i-1},\ u_i,\ u_{i+1}$ is rather low and it can be neglected. This is what is going to be done.

Realizations can $u_{i-1},\ u_i,\ u_{i+1}$ be set in random order. There are can be only *six* options for the location of these points. In *four* of them extreme point appears. It means that probability of extreme points in arbitrary sequence of three values $u_{i-1},\ u_i,\ u_{i+1}$ equals to 2/3. So,

$$M(\pi_u) = 2(n-2)/3 .$$

Variance $D(\pi_u)$ can be calculated in the following way. We have

$$D(\pi_u) = M(\pi_u{}^2) - [M(\pi_u)]^2 .$$

$$M(\pi_u{}^2) = M\left[\left(\sum_{i=1}^{n-2} x_i\right)^2\right] = M[\sum_{n-2} x_i^2 + 2\sum_{n-3} x_i x_{i+1} + 2\sum_{n-4} x_i x_{i+2} + \sum_{(n-4)(n-5)} x_i x_{i+j}],\ j > 2.$$

The number of terms in each of the four sums is indicated by the indexunder the symbol of the sum and totally equals to $(n-2)^2$. In fact,

$$n-2+2(n-3)+2(n-4)+(n-4)(n-5) = (n-2)^2 .$$

Decomposition $M(\pi_u{}^2)$ by sums of squares and mixed products of values for $j = 0, 1, 2, \ldots$ is convenient for further calculations.

Let's calculate consequently values

$$M(x_i^2),\ M(x_i x_{i+1}),\ M(x_i x_{i+2}),\ M(x_i x_{i+j}),\ j>3.$$

From equality $x_i^2 = x_i$ it follows that $M(x_i^2) = 2/3$.

To calculate $M(x_i x_{i+1})$ it is necessary to consider four consecutive values: $u_{i-1}, u_i, u_{i+1}, u_{i+2}$. Reasoning in the same way as in finding $p(x_i = 1)$, we analyze 4! = 24 options of possible orders of magnitude of the values $u_{i-1}, u_i, u_{i+1}, u_{i+2}$. As far as $M(x_i x_{i+1}) = p(x_i x_{i+1} = 1)$, we fix only the options where the extreme points are u_i and u_{i+1}. There are 10 such options. Consequently, $M(x_i x_{i+1}) = 5/12$.

To calculate $M(x_i x_{i+2})$, it is necessary to consider five consecutive observations $u_j, j = i-1, i, i+1, i+3, i+3$ and analyze5! = 120 options of arrangement of values u_j, extract those for which points u_i, u_{i+2} are extreme ones. There are 54 such options. Consequently, $M(x_i x_{i+2}) = 9/20.$

In order to calculate $M(x_i x_{i+j})$ for $j \geq 3$, it should be noted that $x_i = g(u_{i-1}, u_i, u_{i+1})$ and $x_{i+j} = g(u_{i+j-1}, u_{i+j} u_{i+j+1})$, that is there are random values x_i, x_{i+j} which are independent. It means that, $M(x_i x_{i+j}) = M(x_i) M(x_{i+j}) = 4/9,\ j > 4$ Thus,

$$D(\pi_u) = \frac{2}{3}(n-2) + 2 \times \frac{5}{12}(n-3) + 2 \times \frac{9}{20}(n-4) + \frac{4}{9}(n-4)(n-5) - \frac{4}{9}(n-2)^2 = \frac{16n-29}{90},$$

which was to be proved.

Let us consider one more (analytical) method of finding $M(\pi_u)$. It essentially makes use of the assumption of the existence of a continuous density function for u_i.

We introduce the vector random variable B$\vec{\zeta} = (\tilde{x}, \tilde{y}, \tilde{z},)$, components of which are independent and each component of each is arranged by uniform law with density function f on the set [0,1]³. Such an area (unit cube) is taken only for reasons of simplicity of the calculations.Then

$$\int_{[0,1]^3} f d\zeta = 1,$$

Where ζ –variable of integration of a random value $\vec{\zeta}$.

As it has been noted before there can be six options ofthe arrangement of the realizations of the values $x, y, \tilde{z}$:

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$$x > y > z,\ x < y < z,\ (x > y < z)\,\&\,(x > z),\ (x < y > z)\,\&\,(x > z),$$
$$(x > y < z)\,\&\,(x < z),\ (x < y > z)\,\&\,(x < z).$$

Let us find the probability of realizing each arrangement. For example for the case $x > y > z$ we have

$$\iiint\limits_{\substack{x>y\\ y>z}} dxdydz = \int_0^1\left(\int_0^x\left(\int_0^y dz\right)dy\right)dx = \int_0^1\left(\int_0^x ydy\right)dx = \int_0^1 \frac{x^2}{2}dx = \frac{x^3}{6}\bigg|_0^1 = \frac{1}{6}.$$

Probabilities of other locations are calculated similarly. For symmetry reasons, they are also equal to 1/6. Hence we obtain that

$$M(x_i) = 1\times p(x_i = 1) + 0\times p(x_i = 0) = 2/3.$$

For what follows we need a random variable, which is given by

$$\tau = \operatorname{sgn}(\xi-\eta)\operatorname{sgn}(\xi-\zeta),$$

where

$$\operatorname{sgn}(a) = \begin{cases} +1, & a > 0, \\ -1, & a < 0, \end{cases} (a \neq 0);$$

ξ, η, ζ – Identically distributed random variables with a continuous density factor. Then variable random τ has distribution law represented in the table 1.1.

Table 1, the law of distribution of a random variable τ

τ	-1	1
$p(\tau)$	1/3	2/3

This fact follows from consideration of the table 2.

When using random variable τ, we can define random variable π_ξ, which will act as a counter of extreme points:

$$\pi_\xi = \begin{cases} 1, & if\, \xi\, extremumpoint; \\ 0, & otherwise. \end{cases}$$

Corresponding formula has the following form

$$\pi_\xi = \frac{1}{2}(1+\tau).$$

Table 2 Options of the relations between random variables ξ, η, ζ

Relation between ξ, η, ζ	τ	$p(\tau)$
$\xi > \eta, \eta > \zeta$	+1	1/6
$\xi > \zeta, \zeta > \eta$	+1	1/6
$\zeta > \xi, \xi > \eta$	-1	1/6
$\zeta > \eta, \eta > \xi$	+1	1/6
$\eta > \zeta, \zeta > \xi$	+1	1/6
$\eta > \xi, \xi > \zeta$	-1	1/6

The law of distribution for π_ξ is represented in the table 3.

Table 3 The law of distribution of discrete random variable π_ξ

π_ξ	0	1
$p(\pi_\xi)$	1/3	2/3

The provided constructions can be used when programming of algorithms for assessing the level of randomness of relationships and the inter relationships of the socio-economic processes.

4. Properties of a sequence of random variables $\{x_i\}_{i=1}^{n-2}$

Let us investigate the nature of the dependence of random variables in the sequence $\{x_i\}_{i=1}^{n-2}$. Its elements act as values of characteristic function $I_U(.)$ sets of extreme points U, representing the factor u.

Acquainted measure of the dependence of a sequence of random variables is the modulus of the difference between the mathematical expectation of the product of these values and the product of their mathematical expectations [4, p. 383]. In our case we have

$$\alpha_{ij} = \left| M(x_i x_{i+j}) - M(x_i)M(x_{i+j}) \right|.$$

For $j=0,1,2,3$ values $M(x_i x_{i+j})$ have been calculated as above. For these j we have $\alpha_0 = 2/9,\ \alpha_1 = 1/36,\ \alpha_2 = 1/180,\ \alpha_j = 0,\ j>2$. Thus it can be seen that random variables $x_1, x_2, \ldots, x_{n-2}$ are partly dependent.

The following option of determining the partial dependence of a sequence of random variables $\xi_1, \xi_2, \ldots$ will be used. Random variables $\xi_1, \xi_2, \ldots$ can be called *k*-dependent, if $\alpha_j = \begin{cases} \prod_{l=i}^{i+j} b_l, 0 \le j \le k; \\ 0,\ j>k, \end{cases}$, where – there are some positive constants.

According to this definition random variables $x_1, x_2, \ldots, x_{n-2}$ are an example of 2-dependent random variables with parameters $b_i = 2/9,\ b_{i+1} = 1/8,\ b_{i+2} = 1/5,\ b_{i+j} = 0,\ j>2.$

There is

Lemma1. Let $\xi_1, \xi_2, \ldots, \xi_n$ and $\eta_1, \eta_2, \ldots, \eta_n$ – are two sequences of random variables, each of them is *k*-dependent inside its sequence and values from different sequences are pair wise independent. Then random values of the sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$, where $\zeta_i = \xi_i \eta_i, i=1,2,\ldots,n$ are *k*-dependent.

Lemma 1 admits two consecutive generalizations.

Lemma2. Let random variables $\xi_1, \xi_2, \ldots, \xi_n$ are *k*1-dependent.Random variables $\eta_1, \eta_2, \ldots, \eta_n$ are k_2-dependent. And random values $\xi_{i'}$ and $\eta_{i''}$ are l -dependent, $i', i'' \in \{1,2,\ldots,n\}$. Then random variables $\zeta_1, \zeta_2, \ldots, \zeta_n$, where $\zeta_i = \xi_i \eta_i, i=1,2,\ldots,n$, are no more than $\max(k_1, k_2, l)$ -dependent.

Proof. Let $j > \max(k_1, k_2, l)$. Then random values ξ_i, η_i do not depend on random values ξ_{i+j}, η_{i+j}. thus

$$\alpha_j = |M(\zeta_i \zeta_{i+j}) - M(\zeta_i) M(\zeta_{i+j})| = |M(\xi_i \eta_i \xi_{i+j} \eta_{i+j}) - M(\xi_i \eta_i) M(\xi_{i+j} \eta_{i+j})| =$$
$$|M(\xi_i \eta_i) M(\xi_{i+j} \eta_{i+j}) - M(\xi_i \eta_i) M(\xi_{i+j} \eta_{i+j})| = 0.$$

Which was to be proved.

Lemma3. Let random values $\xi_1^p, \xi_2^p, \ldots, \xi_n^p$ are k_p -dependent, $p=1,2,\ldots,P$; random values $\xi_{i'}^{p'}, \xi_{i''}^{p''}$ are $k_{p'p''}$ -dependent $i', i'' \in \{1,2,\ldots,n\}$. Then random values $\zeta_1, \zeta_2, \ldots, \zeta_n$, where $\zeta_i = \prod_{p=1}^{P} \xi_i^p$, are no more than $\max\left[\max_p(k_p), \max_{p',p''}(k_{p'p''})\right]$ -dependent.

This property is proved by the same method as the previous ones.

From the properties of the nature of the dependence of the terms of k -dependent random variables it follows, that with an increase in the number of factors in $\zeta_i = \prod_{p=1}^{P} \xi_i^p$ (that is with an increase of *P*) degree of dependence between random variables (number *k*) does not decrease.

Let us find the probability of the fact that $k+1$ consecutive sequence points $u_0, u_1, ..., u_{n-1}$ $(k \le n-3)$ are extreme.

Case $k=1$ (2-dependence) has already been studied above: $p(x_1 x_2 = 1) = 5/12.$

Let $k=2$.

To determine the number of three extreme points in succession, it is necessary to consider five successive realizations of random variables $u_{i-1}, u_i, u_{i+1}, u_{i+2}, u_{i+3}$. Totally there can be $5! = 120$ sequences of the location of the arrangement of these values. Out of them in 32 cases points u_i, u_{i+1}, u_{i+2} will be extreme ones. So,

$$p(x_i x_{i+1} x_{i+2} = 1) = \frac{4}{15}.$$

Hence the conditional probability of point u_{i+2} being extreme under condition that the previous two points u_i, u_{i+1} are extreme ones, equals to

$$p(x_{i+2} = 1 | x_i x_{i+1} = 1) = \frac{16}{25}.$$

In case $k=3$ we have

$$p(x_i x_{i+1} x_{i+2} x_{i+3} = 1) = p(x_i x_{i+1} x_{i+2} = 1) p(x_{i+3} = 1 | x_i x_{i+1} x_{i+2} = 1).$$

As far random variables x_i 2-dependent, then

$$p(x_{i+3} = 1 | x_i x_{i+1} x_{i+2} = 1) = p(x_{i+3} = 1 | x_{i+1} x_{i+2} = 1).$$

It means,

$$p(x_i x_{i+1} x_{i+2} x_{i+3} = 1) = \frac{4}{15} \times \frac{16}{25} = \frac{64}{375}.$$

By induction we have general formula

$$p(x_i x_{i+1} ... x_{i+k} = 1) = \frac{1}{3} \times \left(\frac{4}{5}\right)^{2(k-1)-1}.$$

In what follows we need a generalization of the Kolmogorov inequality to the case of 2-dependent random variables $x_i, x_2, ..., x_{n-2}$ [5].

Theorem 1. For characteristic function $I_U(u) = x$ there is inequality

$$p\left\{\max_{1\le k\le n-2} \left|\sum_{i=1}^{k} x_i - \frac{2}{3}k\right| < t\left[\frac{16(k+2)-29}{90}\right]^{\frac{1}{2}}\right\} > 1 - t^{-2}.$$

Proof; To prove Kolmogorov in equality is just like proving the inequality $p \le t^{-2}$, where p – is probability that at least one of the inequalities is accomplished

$$\left|\sum_{i=1}^{k} x_i - \frac{2}{3}k\right| < t\left[\frac{16(k+2)-29}{90}\right]^{\frac{1}{2}}, k = 1, 2, ..., n-2.$$

Let the random variables $Y_k, k = 1, 2, ..., n-2$, be so that $Y_\nu = 1$, if the inequalities are accomplished

$$\left|\sum_{i=1}^{\nu} x_i - \frac{2}{3}\nu\right| \ge t\left[\frac{16(\nu+2)-29}{90}\right]^{\frac{1}{2}},$$

$$\left|\sum_{i=1}^{k} x_i - \frac{2}{3}k\right| < t\left[\frac{16(k+2)-29}{90}\right]^{\frac{1}{2}}, \text{ для } k = 1, 2, ..., \nu-1.$$

$Y_\nu = 0$ in all the other cases. Then

$$p = P\{Y_1 + Y_2 + ... + Y_{n-2} = 1\}.$$

As far as $\sum_{k=1}^{n-2} Y_k$ equals to 0 or 1, then the inequality is always accomplished

$$\sum_{k=1}^{n-2} Y_k \le 1.$$

Hence

$$\sum_{k=1}^{n-2} M\left\{Y_k\left[\sum_{i=1}^{n-2} x_i - \frac{2}{3}(n-2)\right]^2\right\} \le \frac{16n-29}{90}. \quad (4)$$

Let us estimate the terms of the left-hand side of the in equality (4). For this we set

$$U_k = \left(\sum_{i=1}^{n-2} x_i - \frac{2}{3}(n-2)\right) - \left(\sum_{i=1}^{k} x_i - \frac{2}{3}k\right) = \sum_{\nu=k+1}^{n-2}(x_\nu - \frac{2}{3}).$$

Then

$$M\left\{Y_k\left[\sum_{i=1}^{n-2}x_i-\frac{2}{3}(n-2)\right]^2\right\}=M\left[Y_k\left(\sum_{i=1}^{k}x_i-\frac{2}{3}k\right)^2\right]+ \quad (5)$$
$$2M\left[Y_kU_k\left(\sum_{i=1}^{k}x_i-\frac{2}{3}k\right)\right]+M\left(Y_kU_k{}^2\right).$$

Next, we need to prove that the second term of (5) is non negative. The property of the 2-dependence of random variables creates additional difficulties in proving this fact. Further we have

$$M\left[Y_kU_k\left(\sum_{\nu=1}^{k}x_\nu-\frac{2}{3}k\right)\right]=M\left[Y_k\sum_{\nu=k+1}^{n-2}(x_\nu-\frac{2}{3})\sum_{\nu=1}^{k}(x_\nu-\frac{2}{3})\right]=$$
$$M\left[Y_k\left(\sum_{\nu=1}^{k-3}(x_\nu-\frac{2}{3})+(x_{k-2}-\frac{2}{3})+(x_{k-1}-\frac{2}{3})+(x_k-\frac{2}{3})\right)\times\right.$$
$$\left.\left((x_{k+1}-\frac{2}{3})+(x_{k+2}-\frac{2}{3})+\sum_{\nu=k+3}^{n-2}(x_\nu-\frac{2}{3})\right)\right]=$$

$$M\left[Y_k\left(\sum_{\nu=1}^{k-3}(x_\nu-\frac{2}{3})(x_{k+1}-\frac{2}{3})+\sum_{\nu=1}^{k-3}(x_\nu-\frac{2}{3})(x_{k+2}-\frac{2}{3})+\right.\right.$$
$$\sum_{\nu=1}^{k-3}(x_\nu-\frac{2}{3})\sum_{\nu=k+3}^{n-2}(x_\nu-\frac{2}{3})+(x_{k-2}-\frac{2}{3})(x_{k+1}-\frac{2}{3})+$$
$$(x_{k-2}-\frac{2}{3})(x_{k+2}-\frac{2}{3})+(x_{k-2}-\frac{2}{3})\sum_{\nu=k+3}^{n-2}(x_\nu-\frac{2}{3})+$$
$$(x_{k-1}-\frac{2}{3})(x_{k+1}-\frac{2}{3})+(x_{k-1}-\frac{2}{3})(x_{k+2}-\frac{2}{3})+$$
$$(x_{k-1}-\frac{2}{3})\sum_{\nu=k+3}^{n-2}(x_\nu-\frac{2}{3})+(x_k-\frac{2}{3})(x_{k+1}-\frac{2}{3})+$$
$$\left.\left.(x_k-\frac{2}{3})(x_{k+2}-\frac{2}{3})+(x_k-\frac{2}{3})\sum_{\nu=k+2}^{n-2}(x_\nu-\frac{2}{3})\right)\right]=$$

$$\sum_{\nu=1}^{k-3} M(Y_k x_\nu x_{k+1}) - \frac{2}{3}\sum_{\nu=1}^{k-3} M(Y_k x_\nu) - \frac{2}{3}(k-3)M(Y_k x_{k+1}) +$$

$$\frac{4}{9}(k-3)M(Y_k) + \sum_{\nu=1}^{k-3} M(Y_k x_\nu x_{k+2}) - \frac{2}{3}\sum_{\nu=1}^{k-3} M(Y_k x_\nu) -$$

$$\frac{2}{3}(k-3)M(Y_k x_{k+2}) + \frac{4}{9}(k-3)M(Y_k) + M(Y_k x_{k-2} x_{k+1}) -$$

$$\frac{2}{3}M(Y_k x_{k-2}) - \frac{2}{3}M(Y_k x_{k+1}) + \frac{4}{9}M(Y_k) + M(Y_k x_{k-2} x_{k+2}) -$$

$$\frac{2}{3}M(Y_k x_{k-2}) - \frac{2}{3}M(Y_k x_{k+2}) + \frac{4}{9}M(Y_k) + M(Y_k x_{k-1} x_{k+1}) -$$

$$\frac{2}{3}M(Y_k x_{k-1}) - \frac{2}{3}M(Y_k x_{k+1}) + \frac{4}{9}M(Y_k) + M(Y_k x_{k-1} x_{k+2}) -$$

$$\frac{2}{3}M(Y_k x_{k-1}) - \frac{2}{3}M(Y_k x_{k+2}) + \frac{4}{9}M(Y_k) + M(Y_k x_k x_{k+1}) -$$

$$\frac{2}{3}M(Y_k x_k) - \frac{2}{3}M(Y_k x_{k+1}) + \frac{4}{9}M(Y_k) + M(Y_k x_k x_{k+2}) -$$

$$\frac{2}{3}M(Y_k x_k) - \frac{2}{3}M(Y_k x_{k+2}) + \frac{4}{9}M(Y_k) =$$

$$\sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+1}) - \frac{4}{3}\sum_{\nu=1}^{k} M(Y_k x_\nu) - \frac{2}{3}kM(Y_k x_{k+1}) -$$

$$\frac{2}{3}kM(Y_k x_{k+2}) + \sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+2}) + \frac{8}{9}kM(Y_k).$$

In this way,

$$M\left[Y_k U_k\left(\sum_{\nu=1}^{k} x_\nu - \frac{2}{3}k\right)\right] = M\left[Y_k \sum_{\nu=k+1}^{n-2}(x_\nu - \frac{2}{3})\sum_{\nu=1}^{k}(x_\nu - \frac{2}{3})\right] =$$

$$\sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+1}) - \frac{4}{3}\sum_{\nu=1}^{k} M(Y_k x_\nu) - \frac{2}{3}kM(Y_k x_{k+1}) -$$

$$\frac{2}{3}kM(Y_k x_{k+2})+\sum_{\nu=1}^{k}M(Y_k x_\nu x_{k+2})+\frac{8}{9}kM(Y_k).$$

Let's consider each term separately. Using the generalization of the conditional mathematical expectation property for the case of a vector argument and the fact that $Y_k = f(x_1, x_2, ..., x_k)$, we have

$$M(Y_k x_\nu x_{k+1}) = M[Y_k M(x_\nu x_{k+1} \mid x_1, x_2, ..., x_k)]$$

for $\nu = 1, 2, ..., k-2.$

Let us calculate the mathematical expectation $M(x_\nu x_{k+1} \mid x_1, x_2, ..., x_k)$ for $\nu = 1, 2, ..., k-2.$

$$M(x_\nu x_{k+1} \mid x_1, x_2, ..., x_k) = p(x_\nu x_{k+1} \mid x_1 x_2 ... x_k = 1) =$$

$$p(x_\nu x_{k+1} \mid x_{\nu-2} x_{\nu-1} x_{\nu+1} x_{\nu+2} x_{k-1} x_k = 1) =$$

$$\frac{p(x_{\nu-2} x_{\nu-1} x_\nu x_{\nu+1} x_{\nu+2} x_{k-1} x_k x_{k+1} = 1)}{p(x_{\nu-2} x_{\nu-1} x_{\nu+1} x_{\nu+2} x_{k-1} x_k = 1)} =$$

$$\frac{p(x_{\nu-2} = 1 \mid x_{\nu-1} x_\nu = 1)\, p(x_{\nu-1} x_\nu x_{\nu+1} x_{\nu+2} x_{k-1} x_k x_{k+1} = 1)}{p(x_{\nu-2} = 1 \mid x_{\nu-1} = 1)\, p(x_{\nu-1} x_{\nu+1} x_{\nu+2} x_{k-1} x_k = 1)} =$$

$$\frac{\frac{16}{25}\, p(x_{\nu-1} = 1 \mid x_\nu x_{\nu+1} = 1)\, p(x_\nu x_{\nu+1} x_{\nu+2} x_{k-1} x_k x_{k+1} = 1)}{\frac{5}{12}\frac{3}{2}\, p(x_{\nu-1} = 1 \mid x_{\nu+1} = 1)\, p(x_{\nu+1} x_{\nu+2} x_{k-1} x_k = 1)} =$$

$$\frac{\frac{16}{25}\frac{16}{25}\, p(x_\nu = 1 \mid x_{\nu+1} x_{\nu+2} = 1)\, p(x_{\nu+1} x_{\nu+2} x_{k-1} x_k x_{k+1} = 1)}{\frac{5}{12}\frac{3}{2}\frac{9}{20}\frac{3}{2}\, p(x_{\nu+1} = 1 \mid x_{\nu+2} = 1)\, p(x_{\nu+2} x_{k-1} x_k = 1)} =$$

$$\frac{\left(\frac{16}{25}\right)^3 p(x_{\nu+1} = 1 \mid x_{\nu+2} = 1)\, p(x_{\nu+2} x_{k-1} x_k x_{k+1} = 1)}{\frac{5}{12}\frac{3}{2}\frac{9}{20}\frac{3}{2}\frac{5}{12}\frac{3}{2}\, p(x_{\nu+2} = 1 \mid x_{\nu+2} = 1)\, p(x_{k-1} x_k = 1)} =$$

$$\frac{\left(\frac{16}{25}\right)^3 \frac{5}{12}\frac{3}{2}\frac{2}{3}\frac{4}{15}}{\frac{5}{12}\frac{3}{2}\frac{9}{20}\frac{3}{2}\frac{5}{12}\frac{3}{2}\frac{2}{3}\frac{5}{12}} = \left(\frac{16}{25}\right)^4 \frac{64}{27}.$$

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For $\nu = k-1$ and $\nu = k$, correspondingly, we have

$$M(x_{k-1}x_{k+1} \mid x_1 \ldots x_k) = \left(\frac{16}{25}\right)^3 \frac{40}{27}, M(x_k x_{k+1} \mid x_1 \ldots x_k) = \left(\frac{16}{25}\right)^2.$$

That is why

$$\sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+1}) = \left[(k-2)\left(\frac{16}{25}\right)^4 \frac{64}{27} + \left(\frac{16}{25}\right)^3 \frac{40}{27} + \left(\frac{16}{25}\right)^2\right] \times M(Y_k).$$

Further

$$\sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+2}) = M[Y_k M(x_\nu x_{k+2} \mid x_1 \ldots x_k)].$$

For $\nu = 1, 2, \ldots, k-2$ we have

$$M(x_\nu x_{k+2} \mid x_1 \ldots x_k) = M(x_\nu x_{k+2} \mid x_{\nu-2} x_{\nu-1} x_{\nu+1} x_{\nu+2} x_k) =$$

$$p(x_\nu x_{k+2} = 1 \mid x_{\nu-2} x_{\nu-1} x_{\nu+1} x_{\nu+2} x_k = 1) =$$

$$\frac{p(x_{\nu-2} x_{\nu-1} x_\nu x_{\nu+1} x_{\nu+2} x_k x_{k+2} = 1)}{p(x_{\nu-2} x_{\nu-1} x_{\nu+1} x_{\nu+2} x_k = 1)} = \left(\frac{16}{25}\right)^3 \frac{8}{5}.$$

For $\nu = k-1$ and $\nu = k$ we have correspondingly

$$M(x_{k-1} x_{k+2} \mid x_{k-3} x_{k-2} x_k) = \left(\frac{16}{25}\right)^2,$$

$$M(x_k x_{k+2} \mid x_{k-2} x_{k-1}) = \left(\frac{16}{25}\right) \frac{27}{40}.$$

That is why

$$\sum_{\nu=1}^{k} M(Y_k x_\nu x_{k+2}) = \left[(k-2)\left(\frac{16}{25}\right)^3 \frac{5}{2} + \left(\frac{16}{25}\right)^2 + \frac{16}{25}\frac{27}{40}\right] \times M(Y_k).$$

The second term has the following form

$$\frac{4}{3}\sum_{\nu=1}^{k} M(Y_k x_\nu) = \frac{4}{3}\left[(k-2)\left(\frac{16}{25}\right)^3 \frac{64}{27} + \left(\frac{16}{25}\right)^2 \frac{40}{27} + \frac{16}{25}\right] \times M(Y_k).$$

Finally,

$$M(Y_k U_k(\sum_{\nu=1}^{k} x_\nu - \frac{2}{3}k)) =$$

$$\left[(k-2)\left(\frac{16}{25}\right)^4 \frac{64}{27} + \left(\frac{16}{25}\right)^3 \frac{40}{27} + \left(\frac{16}{25}\right)^2\right] \times M(Y_k) -$$

$$\frac{4}{3}\left[(k-2)\left(\frac{16}{25}\right)^3 \frac{64}{27} + \left(\frac{16}{25}\right)^2 \frac{40}{27} + \frac{16}{25}\right] \times M(Y_k) -$$

$$\frac{2}{3}\frac{16}{25} kM(Y_k) - \frac{2}{3}\frac{27}{40} kM(Y_k) +$$

$$\left[(k-2)\left(\frac{16}{25}\right)^3 \frac{5}{2} + \left(\frac{16}{25}\right)^2 + \frac{16}{25}\frac{27}{40}\right] \times M(Y_k) + \frac{8}{9} kM(Y_k) =$$

$$(0,236k - 0,708) M(Y_k).$$

Thus starting with the $k = 3$, the quantity

$$M(Y_k U_k(\sum_{\nu=1}^{k} x_\nu - \frac{2}{3}k))$$

is nonnegative. From this and from (5) we have

$$M\left\{Y_k\left[\sum_{i=1}^{n-2} x_i - \frac{2}{3}(n-2)\right]^2\right\} \geq M\left[Y_k\left[\sum_{i=1}^{k} x_i - \frac{2}{3}(n-2)\right]^2\right]. \quad (6)$$

Random variable $Y_k \neq 0$ is only when

$$\left|\sum_{i=1}^{k} x_i - \frac{2}{3}k\right| \geq t\left[\frac{16(k+2)-29}{90}\right]^{\frac{1}{2}}.$$

That is why

$$Y_k\left(\sum_{i=1}^{k} x_i - \frac{2}{3}k\right)^2 \geq t^2 \frac{16(k+2)-29}{90} Y_k.$$

Comparing inequalities (4) and (6), we have

$$\frac{16n-29}{90} \geq t^2 \frac{16n-29}{90} M(Y_1+Y_2+\ldots+Y_{n-2}).$$

Hence taking into account that

$$M\left(\sum_{i=1}^{n-2} Y_i\right) = P\left(\sum_{i=1}^{n-2} Y_i = 1\right) = p,$$

We have $pt^2 \leq 1$.

Generalization of Kolmogorov inequality has been proved.

Next we estimate the law of distribution π_U of extreme points of the sequence $u_0, u_1, \ldots, u_{n-1}$. To do this we pass to the random variables

$\tilde{x}_i = x_i - M(x_i) = x_i - \frac{2}{3}, i = 1, 2, \ldots, n-2$, and use the theorem 19.1.1 [4], which states that if $\{\ldots, x_{-1}, x_0, x_1, \ldots\}$ is stationary sequence of *m*-dependent random variables then

$$\sigma^2 = M(x_0{}^2) + 2\sum_{j=1}^{\infty} M(x_0 x_j) < \infty$$

And if $\sigma^2 \neq 0$, then

$$\lim_{n\to\infty} p\left\{\frac{1}{\sigma\sqrt{n}} \sum_{j=1}^{n} x_j < z\right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2\sigma}} du.$$

The conditions of this theorem for random variables $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-2}$ are obviously satisfied. Consequently, with increasing of n the distribution of the number of extreme points π_U divided by $\sigma(\pi_u)\sqrt{n}$, asymptotically tends to normal.

DISCUSSION & CONCLUSION

In the first paragraph of the article concept and properties of extreme points of time series have been considered in detail. Properties of the partial dependence of extreme points and their generalization have been identified. Generalization of Kolmogorov inequality has been achieved. Asymptotic normal distribution of partially dependant random variables has been proven. This research can benefit the area of social and financial economics, as often they inherently include stochastic factors by their nature, and this modelling approach can be applied.

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