Regular, Normal and Tychonoff Spaces of \( fg^{##} \) –closed set in Fuzzy Topological Spaces

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**ABSTRACT:** In this paper, as an application of fuzzy \( g^{##} \) - closed set in fuzzy topological spaces (fts), which is a class of closed fuzzy set. A fuzzy \( g^{##} \) - closed set is obtained by generalizing \( g^{##} \)- open fuzzy sets via fuzzy \( \alpha \)-closure. We introduced and investigated some new type of separation axioms namely, \( fg^{##}\)-Regular Space, \( fg^{##} \)- Tychonoff space, \( fg^{##}\)-Normal spaces, \( fg^{##}\)- Completely normal space in fts. The separation axiom provides a framework where the spaces have enough of continuous mappings. Also we have investigated some of their properties with counter examples.

**KEYWORDS:** \( fg^{##}\)-Space, \( fg^{##}\)-Regular Space, \( fg^{##}\)- Tychonoff space, \( fg^{##}\)-Normal spaces, \( fg^{##}\)- Completely normal space

1. **INTRODUCTION**

The theory of ‘fuzzy subset’ is introduced by Prof. L.A. Zadeh[1] in 1965. It provides a structure to generalize the concepts of general topology into Fuzzy topological spaces (fts). In fuzzy topology a fts is a unpredictable object. On one extreme there are indiscrete spaces with only two open sets. Every set is open in a discrete space on the other extreme. Among discrete and indiscrete spaces, several spaces are accessible. The theory of continuity is associated with every set and huge number of open sets collectively construct of continuous functions. The separation axiom provide a theory that the spaces under observation have enough number of continuous mappings. Fuzzy separation axioms were defined by Sinha [2] and Bayoumi and Ismail Ibedou [3] introduced a new theory of fuzzy separation axioms. In 1965, as an application of \( \alpha \)-closed set N Jastad [4] introduced \( \alpha \)-spaces. In 1968, Chang[5] introduced the structure of fuzzy topology based on general topology. Lowen [6] investigated the connections among Fuzzy topological spaces and Topological spaces with the help of two function called as \( \bar{w} \) and \( l \). In 1970 N Levine [7] introduced and studied \( g^{##} \) – \( T_{1/2} \) which properly lies between \( T_3 \) and \( T_1 \)-spaces. In 1996 Sostak [8] focused on some concepts of basic structures of fuzzy topology. In 2000, Veerakumar [9] introduced and studied the concept of \( g^{##}\)-closed sets and \( T_{2/2} \)-spaces in general topology. In 2008, Chen et al.[10] introduced and studied \( w\)-convergence theory and its applications. Subsequently many researchers like, Azad [11], Bhaunik et al.[12], Munshi[13], and many others have contributed to the development of fuzzy topological spaces.

In this paper, we introduced and studied fuzzy \( g^{##} - T_1 \) spaces (where \( i = 0, 1, 2, 2 \frac{1}{2}, 3, 3 \frac{1}{2}, 4, 5 \) \( , fg^{##}\)-regular, \( fg^{##}\)-Tychonoff space, \( fg^{##}\)-Normal spaces, \( fg^{##}\)-Completely normal space along with their properties and counter examples. We also considered the fuzzy disjointness that agrees with ordinary set, theoretical disjointness in the crisp case.

2. **PRELIMINARIES**

Throughout this paper \( (X, T), (Y, \sigma) \) and \( (Z, \eta) \) or (simply \( X, Y, \) and \( Z) \) represents non-empty fuzzy topological spaces on which no separation axiom is assumed unless explicitly stated. For a subset \( A \) of a space \( (X, T) \), \( cl(A) \) and \( int(A) \) denotes the closure, interior and the compliment of \( A \) respectively.

Definition 2.01: A fuzzy set \( A \) of a fts \( (X, T) \) is called a \( g^{##}\)-closed set, if \( acl(A) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is \( ag\)-open fuzzy set in \( (X, T) \).[14]

Complement of \( g^{##}\)-closed fuzzy set is called \( g^{##}\)-open fuzzy set

Definition 2.02: Let \( X \) and \( Y \) be two fuzzy topological Spaces, A function \( f: X \to Y \) is called a \( g^{##}\)-continuous (strongly \( fg^{##}\)-continuous) if \( f^{-1}(V) \) is \( g^{##}\)-open fuzzy set in \( X \), for every \( ag\)-open set in \( Y \).[14]

Definition 2.03: A space \( X \) is said to be:

1) a fuzzy \( T_{0\alpha} \)-space, if for each pair of distinct points \( x, y \) of \( X \), there exists a open set containing one but not the other.[9]

2) a fuzzy \( T_{1\alpha} \)-space, if for each pair of distinct points \( x, y \) of \( X \), there exists a pair of open sets, one containing \( x \) but not \( y \) and the containing \( y \) but not \( x \).[9]

3) a fuzzy \( T_{2\alpha} \)-space, if for each pair of distinct points \( x, y \) of \( X \), there exists a pair of open sets, one containing \( x \) and the other containing \( y \).[9]

4) a regular space, if for every closed set \( G \) and a point \( x \in G \), there exist disjoint open sets \( U \) and \( V \) such that \( G \subseteq U \) and \( x \in V \).[9]

5) A normal space, if for every pair of disjoint closed sets \( F \) and \( G \) of \( X \), there exist disjoint open sets \( U \) and \( V \) such that \( F \subseteq U \) and \( G \subseteq V \).[9]
In the connection with further research work, we considered the definitions of spaces namely, \( f g^{\#}\text{-}T_0 \) Space, \( f g^{\#}\text{-}T_1 \) Space, \( f g^{\#}\text{-}T_2 \) Space, \( f g^{\#}\text{-}T_3 \) Space, \( f g^{\#}\text{-}T_{31} \) Space, \( f g^{\#}\text{-}T_{32} \) Space,
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**Definition 2.04:** A fuzzy topological space \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}T_0 \) space (briefly \( f g^{\#}\text{-}T_0 \) Space), if every \( g^{\#}\text{-}\)open fuzzy set is open and if every \( g^{\#}\text{-}\)closed fuzzy set is closed. \([15]\)

**Definition 2.05:** A fuzzy topological space \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}T_1 \) space (briefly \( f g^{\#}\text{-}T_1 \) Space), if for each pair of distinct points \( x,y \in X \) there exist \( g^{\#}\text{-}\)open fuzzy sets \( U \) and \( V \) such that \( U(x) = 1 \) but \( U(y) = 0 \) and \( V(x) = 0 \) but \( V(y) = 1 \). \([15]\)

**Definition 2.06:** A fuzzy topological space \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}T_2 \) space (briefly \( f g^{\#}\text{-}T_2 \) Space), if every \( g^{\#}\text{-}\)closed fuzzy set is \( (a-c) \)-closed fuzzy set in \((X,T)\). \([15]\)

**Definition 2.07:** A fts \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}T_3 \) space (briefly \( f g^{\#}\text{-}T_3 \) Space), if for each pair of distinct points \( x,y \in X \), there exist \( g^{\#}\text{-}\)open fuzzy sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( U(x) = 1 \) and \( V(x) = 1 \) and \( U \land V = 0 \). \([15]\)

**Definition 2.08:** A fts \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}T_{31} \) space (briefly \( f g^{\#}\text{-}T_{31} \) Space), if for each pair of distinct points \( x,y \in X \), there exist two disjoint \( g^{\#}\text{-} \)open fuzzy sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( U(x) = 1 \) and \( V(y) = 1 \) and \( cl(U) \land cl(V) = 0 \). \([15]\)

**3. \( f g^{\#}\text{-}\)REGULAR SPACE IN FUZZY TOPOLOGICAL SPACES**

In continuation with above definitions, we introduced some new type of separation axiom in fuzzy topological spaces called \( f g^{\#}\text{-}\)Regular Space, along with their properties with counter examples.

**Definition 3.01:** A fts \((X,T)\) is said to be regular, if for each \( x \in X \) and a \( g^{\#}\text{-}\)closed fuzzy set \( A \) with \( A(x) = 0 \), there exist open fuzzy sets \( U \) and \( V \) such that \( U(x) = 1 \), \( A \subseteq V \) and \( U \subseteq 1 - V \).

**Definition 3.02:** A fts \((X,T)\) is said to be fuzzy \( g^{\#}\text{-}\)regular (briefly \( f g^{\#}\text{-}\)regular), if for each \( x \in X \) and a \( g^{\#}\text{-}\)closed fuzzy set \( A \) with \( A(x) = 0 \), there exist open fuzzy sets \( U \) and \( V \) such that \( U(x) = 1 \), \( A \subseteq V \) and \( U \subseteq 1 - V \).

**Theorem 3.03:** Every \( g^{\#}\text{-}\)regular space in fts \((X,T)\) is regular space in \( X \).

**Proof:** Let \( X \) be a \( g^{\#}\text{-}\)regular space in fts \((X,T)\). Let \( x \in X \) and \( A \) be a \( g^{\#}\text{-}\)closed fuzzy set in \( X \) with \( A(x) = 0 \). Then \( A \) is \( g^{\#}\text{-}\)closed fuzzy set in \( X \). Since \( X \) is \( g^{\#}\text{-}\)regular in fts \((X,T)\), there exist open fuzzy sets \( U \) and \( V \) such that \( U(x) = 1 \), \( A \subseteq V \) and \( U \subseteq 1 - V \). Hence \( X \) is regular space in \((X,T)\).

The converse of the above theorem need not be true, as seen from the following examples.

**Example 3.04:** Let \( X = \{a,b,c\} \) the \( A,B,C \) and \( D \) be fuzzy sets, where \( A = \{(a), (b), (c)\}, \ B = \{(a), (b), (c)\}, \ C = \{(a), (b), (c)\}, \ D = \{(a), (b), (c)\} \). Then \((X,T)\) is fuzzy topological space with \( T = \{0,1,\{A\},\{C\}\}\). Then \((X,T)\) is fuzzy regular space and \( f g^{\#}\text{-}\)space in fts \((X,T)\).

**Proof:** Omitted.

**Theorem 3.05:** If \( X \) is regular space and \( f g^{\#}\text{-}\)space in fts \((X,T)\) is \( g^{\#}\text{-}\)regular space in fts \((X,T)\).

**Proof:** Omitted.

**Theorem 3.06:** If a fts \((X,T)\) is regular and a fuzzy \( g^{\#}\text{-}T_2 \) space. Then \((X,T)\) is \( g^{\#}\text{-}\)regular space in fts \((X,T)\).

**Proof:** Omitted.

**Theorem 3.07:** The following properties are equivalent:

a. \( X \) is \( g^{\#}\text{-}\)regular space in fts \((X,T)\).

b. For each \( x \in X \) and \( g^{\#}\text{-}\)open fuzzy set in \( U \) with \( U(x) = 1 \), there exists open sets \( V \) with \( V(x) = 1 \), such that \( V \subseteq cl(V) \subseteq U \).

c. For each \( x \in X \) and \( A \) be a \( g^{\#}\text{-}\)closed fuzzy set in \( X \) with \( A(x) = 0 \) and an open fuzzy set \( V \) with \( V(x) = 1 \), such that \( A \subseteq 1 - cl(V) \).

**Theorem 3.08:** A fuzzy subspace of a \( g^{\#}\text{-}\)regular space in fts \((X,T)\) is \( g^{\#}\text{-}\)regular space.

**Proof:** Omitted.

**Theorem 3.09:** If \( f \colon X \rightarrow Y \) is an open, \( f g^{\#}\text{-}\)irresolute bijective and \( X \) is \( g^{\#}\text{-}\)regular fts. Then \( Y \) is \( g^{\#}\text{-}\)regular fts.

**Proof:** Let \( y \in Y \) an \( \alpha \) \( A \) be a \( g^{\#}\text{-}\)closed fuzzy set in \( Y \). Since \( f \) is \( f g^{\#}\text{-}\)irresolute, \( f^{-1}(A) \) is \( g^{\#}\text{-}\)closed fuzzy set in \( X \). Subtract \( f(x) = y \), then \( 1 - f^{-1}(A)(x) \) is \( g^{\#}\text{-}\)closed fuzzy set in \( X \). Since \( X \) is \( g^{\#}\text{-}\)regular fts, there exist open fuzzy sets \( \forall \), \( V \) such that \( U(x) = 1 \), \( f^{-1}(A) \subseteq V \) and \( U \subseteq 1 - V \). Since \( f \) is open and \( \alpha \)-bijective, we have \( f(U) \subseteq \alpha \) \( V \) and \( f(U) \subseteq 1 - f(V) \). Hence \( Y \) is \( g^{\#}\text{-}\)regular.

**4. \( f g^{\#}\text{-}\)NORMAL SPACES IN FTS**

In this section, we introduced some new type of separation axiom in fuzzy topological spaces called \( f g^{\#}\text{-}\)Normal spaces, \( f g^{\#}\text{-}T_3 \) and \( f g^{\#}\text{-}T_4 \) Spaces. Also investigated their properties with counter examples.

**Definition 4.01:** A fts \((X,T)\) is said to be normal, if for every closed fuzzy set \( F \) and an open fuzzy set \( G \) such that \( F \subseteq G \), there exist a fuzzy set \( A \), such that \( F \subseteq int(A) \leq cl(A) \subseteq G \).

**Definition 4.02:** A fts \((X,T)\) is said to be \( g^{\#}\text{-}\)normal, if for every \( g^{\#}\text{-}\)closed fuzzy set \( F \) and an \( g^{\#}\text{-}\)open fuzzy set \( G \) such that \( F \subseteq G \), there exist a fuzzy set \( A \), such that \( F \subseteq int(A) \leq cl(A) \subseteq G \).

**Theorem 4.03:** For a fts \((X,T)\), the properties are equivalent:

a. \( X \) is a \( g^{\#}\text{-}\)normal fts.

b. For any two \( g^{\#}\text{-}\)closed fuzzy sets \( A \) and \( B \) in \( X \) such that \( A \subseteq B \), there exists open sets \( U \) such that \( A \subseteq U \), \( B \subseteq V \) and \( U \subseteq 1 - V \).
Definition 4.11: A fts \((X, T)\) is said to be fuzzy \(g^{**}\)-\(T_3\) space (briefly \(f\) \(g^{**}\)-\(T_3\) space), if it is fuzzy \(g^{**}\)-regular space in \((X,T)\), and also fuzzy \(g^{**}\)-\(T_3\) space.

Theorem 4.12: Every \(f\) \(g^{**}\)-\(T_3\) space is \(f\) \(T_3\) space.

Proof: Let \(X \subset g^{**}\)-\(T_3\) space in \((X,T)\), by definition \(X\) is also \(g^{**}\)-regular. Let \(A \subset g^{**}\)-closed fuzzy set in \(X\), there exist open fuzzy sets \(U, V\) such that \(U(x) = 1, A \subseteq U \cup V\) and \(V(y) = 0, V \setminus U \subseteq 1\). Therefore \(X\) is fuzzy regular and also fuzzy - \(T_3\) space. Hence \(f\) \(g^{**}\)-\(T_3\) Space is \(f\) \(T_3\) Space.

Theorem 4.13: Every \(f\) \(g^{**}\)-\(T_3\) Space is \(f\) \(g^{**}\)-\(T_2\) Space.

Proof: Omitted.

Theorem 4.14: Every \(f\) \(g^{**}\)-\(T_3\) Space is \(f\) \(g^{**}\)-\(T_2\) Space.

Proof: Omitted.

\(f\) \(g^{**}\)-\(T_4\) SPACE

Definition 4.15: A fts \((X, T)\) is said to be fuzzy - \(T_4\) space (briefly \(f\) \(T_4\)-Space), if it is fuzzy normal space in \((X,T)\), and also fuzzy - \(T_3\) space.

Definition 4.16: A fts \((X, T)\) is said to be fuzzy \(g^{**}\)-\(T_4\) space (briefly \(f\) \(g^{**}\)-\(T_4\) space), if it is fuzzy \(g^{**}\)-normal space in \((X,T)\), and also fuzzy \(g^{**}\)-\(T_3\) space.

Theorem 4.17: Every fuzzy - \(T_4\) Space is \(f\) \(g^{**}\)-\(T_4\) Space.

Proof: Omitted.

Theorem 4.18: Every \(f\) \(g^{**}\)-\(T_4\) Space is \(f\) \(g^{**}\)-\(T_3\) Space.

Proof: Omitted.

5. \(f\) \(g^{**}\)- COMPLETELY REGULAR SPACE (\(f\) \(g^{**}\)-Tychonoff SPACE) IN FTS

In this section, we introduced some new type of separation axiom in fuzzy topological spaces called \(f\) \(g^{**}\)-Tychonoff spaces, \(f\) \(g^{**}\)-completely normal and \(f\) \(g^{**}\)-\(T_3\) Space. Also investigated their properties with counter examples.

Definition 5.01: A fts \((X, T)\) is said to be fuzzy \(g^{**}\)-completely regular (briefly \(f\) \(g^{**}\)-completely regular), if for every \(f\) \(g^{**}\)-closed subset \(A\) of \(X\) and every point \(x \in X \setminus A\), there exist a continuous map \(f: X \rightarrow [0, 1]\) such that \(f(x) = 0\) and \(f(A) = \{1\}\).

Definition 5.02: A \(f\) \(g^{**}\)-Tychonoff space (\(f\) \(g^{**}\)-\(T_3\) space) is \(f\) \(g^{**}\)-completely regular and also \(f\) \(g^{**}\)-\(T_3\) space.

Theorem 5.03: Every \(f\) \(g^{**}\)-Tychonoff Space is a \(f\) \(g^{**}\)-\(T_3\) space.

Proof: Omitted.

\(f\) \(g^{**}\)- COMPLETLY NORMAL SPACE

Definition 5.04: A fts \((X, T)\) is said to be fuzzy \(g^{**}\)-completely normal (briefly \(f\) \(g^{**}\)-completely normal), if for any two \(f\) \(g^{**}\)-separated fuzzy subsets \(A\) and \(B\) of \(X\), there exist open sets \(U, V\) such that \(A \subseteq U, B \subseteq V\) and \(U \cup V = X\).

Definition 5.05: A fts \((X, T)\) is said to be fuzzy \(g^{**}\)-\(T_3\) space (briefly \(f\) \(g^{**}\)-\(T_3\) space), if it is \(f\) \(g^{**}\)-completely normal space and also \(f\) \(g^{**}\)-\(T_3\) space in \((X,T)\).
Theorem 5.06: Every $fg^{\#}$-completely normal space is $fg^{\#}$-normal space and every $fg^{\#} - T_2$ space is $fg^{\#} - T_4$ space.

Proof: Omitted.

Figure 5.07: The Relation between $fg^{\#}$ spaces, where $i = 0, 1, 2, 3, 4, 5$.

6. CONCLUSION

In this paper, we introduced the concept of some separation axioms, such as $fg^{\#}$-Regular Space, $fg^{\#}$-Tychonoff space, $fg^{\#}$-normal space, $fg^{\#}$-Completely normal space in $\mathbb{N}$, and investigated their properties. We also studied relationships between these fuzzy spaces with counter examples. In future, further research will direct to provide a framework where the spaces may obtain enough number of continues mappings.

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REFERENCES