Computing the Geodesic Distance between Two Points in a Polyhedral Solid

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ABSTRACT: This paper presents a method to compute the shortest (geodesic) distance between two given points in a polyhedral solid. The basic difference between the shortest and the geodesic shortest distance is that, in shortest distance between two points in a graph is the shortest path length connecting those two points while geodesic shortest distance between two points is the curvilinear distance measured along the shape (curvature) of the solid. The previous proposed methods are rather complicated, this method is rather simple since it uses square mesh system instead of triangular mesh and also provides a graphical method to calculate the shortest distance. This process can be used to calculate the distance between two points on solids of any size using a suitable scaling factor.

KEYWORDS: Geodesic distance, Polyhedral, Scaling factor.

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1. INTRODUCTION

The conversion of straight line into a curved line is referred to as geodesic line. It was previously referred as the distance between two points in the Earth surface along its curvature. The concept of Geodesic distance is used in mathematics, like measuring the distance between two *nodes* over a sphere in a *graph*. A geodesic line minimizes the distance between two points locally. Equivalently, it is a line formed by a particle when moves with a constant velocity on a surface of any solid. In a plane surface the geodesic line is always a straight line. On a sphere, the geodesic lines are arcs of a circle on the sphere. Geodesics preserve a direction on a surface [1] and have many other interesting properties, *e.g.*, the normal vector at any point of a geodesic arc lies along the normal at that point [1]. Furthermore, no matter how badly a sphere is distorted, there exist an infinite number of closed geodesics arc on it [2]. This general result, demonstrated in the early 1990s, by extending earlier work by Birkhoff [2] who proved in 1917 that there exists at least one closed geodesic on a distorted sphere.

This geodesic line is very important in modern mathematics as well as various other fields. It is used in techniques like computerized brain flattening, texture mapping, surface partitioning, terrain navigation, and path planning [3]. There are previous proclaimed methods like Iterative growth of triangular meshes suggested in [4], method of finding the shortest distance by taking a voxel based approach applied in [5]. In this paper we proposed a simpler method comparative to those previous methods.

2. METHOD

2.1 To reduce the size of the structure to a considerable size

The size of the arbitrary solid might not always be at a considerable and measurable size. In all those cases we multiply or divide the known dimensions by a suitable scaling factor, to make the dimensions measurable. For example, if we want to measure the distance between two places on the Earth, the distance would be in the order of several thousand kilometres which is not very easy to measure, thus we divide all the known dimensions by 1000 or 2000 to make it measurable and then at the time of the final answer we multiply it back to get the correct number.

2.2 To trace out the geodesic line

We have trace out the line or the geodesic distance between the two points. For that we have to introduce an alternate interpretation of the meaning of geodesic line. For that, we will have to consider parallel sun rays falling on the opaque curve. A shadow is obtained on the plane, which is perpendicular to the rays and contains both the points A and B. If the geometric shadow coincides with the original shortest path between the two points, the curve line is said to be the geodesic line. Eventually, that line will turn out to be the geodesic line. In simpler words if we look at the geodesic line directly from above (normally) then we will see the original shortest straight line (not geodesic), or we will see those two lines virtually coinciding, like the sphere in the below Fig 1.

3. CALCULATION OF THE APPROXIMATE GEODESIC

3.1 Formation of square grid mesh system

We divide the surface of the system or the arbitrary solid into a square mesh system. A square mesh system is a continuous network of grids on the surface of the solid which are rectangular in shape and all the vertices being connected to four sides. The solid

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has the geodesic line marked on it (see Fig 2a).
The square mesh is very different from longitude of the Earth since longitudes are wider near the centre or Equator but narrower closed to the poles. In case of square grids, grid boundary lines are equal spacing throughout the sphere.

Each of the rectangular grids is of same sizes and dimensions. We need to choose the size of the rectangular as required very carefully. Smaller the size of each rectangle, more accurate would be our results. In ideal conditions the length of each side of the rectangular tends to zero. We need to carefully choose the size of the each of the rectangle because smaller grid rectangles more difficult and lengthier will be our calculations. Therefore, we need to choose an optimum size. This may lead to a little error or inaccuracy, but the value can be neglected.

3.2 Technique of Surface Flattening

A polyhedral solid is a three dimensional solid which is made of many flat faces and vertices. Each flat surface is a polygon of 'n' number of sides. Examples are tetrahedron, icosahedrons (20 faces) (Fig 2b). Surface Flattening in Geodesics is a technique of conversion of a smooth surface to a flat surface. The main purpose of this in measuring geodesic length is to convert a normal solid into a polyhedral solid.

We will apply this technique of surface flattening in the arbitrary solid to calculate the Geodesic distance. Apply Surface flattening to each of the grid in the above rectangular mesh system, but keeping in mind that the overall curvature of the surface of the solid remains same (constant), for example, if we flatten the surface of each grid of a sphere, at last, the overall structure should still look like a sphere, but with many faces or a polyhedral solid. Like in the Fig 3a, we notice or rather we have to make sure that each segment of the Geodesic line lying on each of the rectangular Grid should now become almost straight instead of the curve nature. This curve line will automatically turn into straight lines when we flatten the surface of each grid. That is a very big advantage for the calculation of the Geodesic distance between the two points. This method of surface flattening converted the arbitrary solid into a polyhedral solid with mane faces, but most importantly the overall curvature of the solid remains same (Like in the Fig 3a).

An important observation is the line passing through the surface of each of the rectangular grid has become a straight line, so the three dimensional curvilinear geodesic line gets converted to a two dimensional straight line whose individual length can be easily found out.

3.3 Calculation of individual length of each line segments

Individual line segment refers to the line segments lying on each of the flattened rectangular surface of the polyhedral solid. For example in the above Fig 3b, we can see each of the five two dimensional line segment in each of the grid. There are five grids involved in this example.

AA_1 is the segment contained in grid number 1; A_1A_2 is contained in grid number 2, ..., A_4A_5 is the line segment in the last grid. All the lines are two dimensional lines and the each of the lines are segments or parts of different line all together (the dotted lines in the Fig 3b).

Now we will consider each of the grids. Grid 1: the individual grid 1 consisting line AA_1 looks like (Fig 4a). In Fig 4a OKCD is the rectangular grid and the line AA_1 is represented in the grid. The dotted lines refer to the original line of the line segment.

Then we have to divide the rectangular graph again in equal parts both horizontally and vertically. For that we first draw two horizontal and two vertical lines in the grid each of whose intersection is the end points of the line segment in that respective grid. In the Fig 4b the lines are RQ, PN which signifies point A and lines AW and CD which signifies the point A1. The main significance of this particular step is to simplify calculations.

In this Fig 4b let WN be x and PQ be y. We did this step so that while making the graph the two concerned points lie on the integer points and not on somewhere middle. Now we have to make the divisions. We consider the point O as origin. We will have to divide the known distances i.e. x and Y by an integer for example 3 or 4. Let them be a and b. This a and b can be different or in rare chances can be same which will give rise to square boxes. a=$\frac{x}{n}$and b=$\frac{y}{m}$ where n and m are integer elements. a and b are the length of each division across X and Y axis. If there is any portion left initially then we can calculate the length in terms of and b. Therefore the created graph looks like Fig 5.

In Fig 5, we can see a = $\frac{x}{m}$ (m = 3 in this case) and b =$\frac{y}{n}$ (n = 6 in this case) and as mentioned above the left out initial portion OW can be represented as 'pa' where p can be any real number, a fraction if less than a and if more than a then we make more vertical lines of width a till we get a fractional remaining. Here we can see that that
$X_1 =\frac{a}{t} + ma$, $Y_1 = \frac{b}{l} + 2\frac{y}{n}$ (coordinate if A_1), $X_2 =\frac{a}{t}$, $Y_2 = \frac{n}{b} + \frac{b}{l} + 2\frac{b}{n}$ (coordinate of A), (The terms $2\frac{y}{n}$ and $2\frac{b}{n}$ are only for this fig (Fig 5), the rest of the terms are general terms applicable for all cases)

Therefore, the distance AA_1 can simply be calculated by distance between two points in two dimensions$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If we know x, we know a, therefore the only thing we have to know is the length of each unit. Therefore this gives the length of the first segment AA_1 = l_1. Similarly we calculate the length of the line segment in second grid = l_2, in third= l_3 ... till n grid = l_n. Therefore the final or Geodesic length of line is given by $l=l_1+l_2+l_3+l_4+...l_n$.

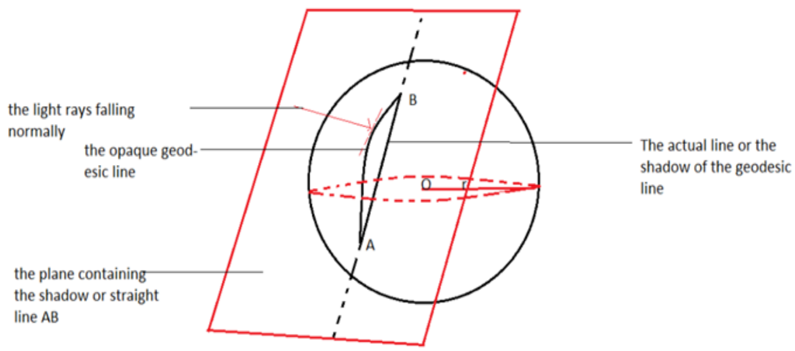

Fig 1: The shadow of the Geodesic falling on the plane.

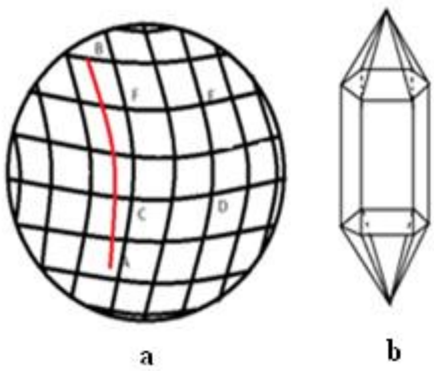

Fig 2: a:- The surface of the solid divided into rectangular mesh system, b:- An Icosahedron.

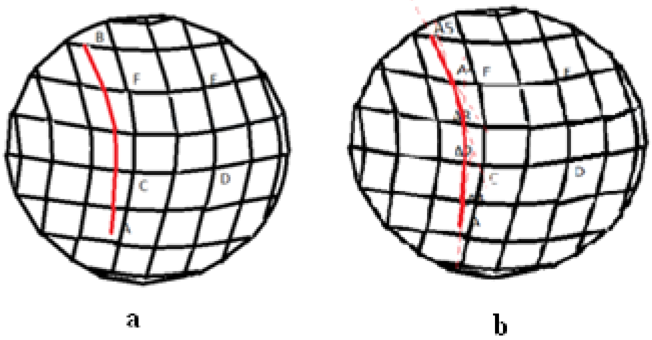

Fig 3: a:- Surface flattening in each rectangular grid in the solid, b:- The line segment of each line in each grid.

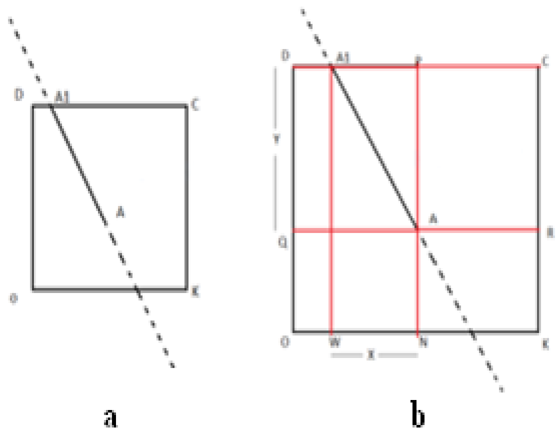

Fig 4: a:- View of each grid, b:- The two points of the line segment.

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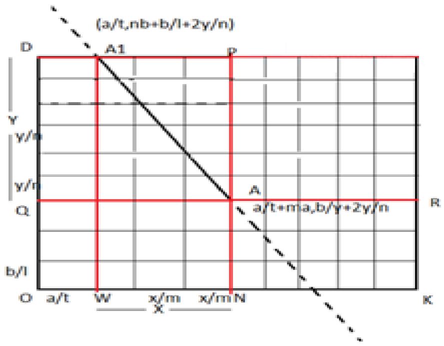

Fig 5: The estimated graph containing the two dimensional line.

3.3 The final length

The final Geodesic length can be calculated by multiplying the above length with the scaling factor we used to scale up or down the big or small body ('k').
L=kl.

4. CONCLUSION

The most important step of this above technique is the tracing the geodesic length by the alternative meaning of geodesic distance. A disadvantage of earlier algorithms was that algorithm was not applicable to convex solid. But the alternative representation for tracing the geodesic length referred in the step two of the above process is applicable for all types of solid including convex since the only thing which we need to know is the geodesic line.

Some problems stays like the process may be simple but it is very lengthy. This process does not give us the exact geodesic length. Since when the surfaces are flattened the very little curve is neglected to a straight line so the length is reduced a bit. So the actual Geodesic length is little bigger than the above calculated length. Geodesic distance may be used in space science in future. According to the rectilinear propagation of light, light always travel in a straight line, it cannot bend. Newton said that this whole space in which light travels is not planar but is curved in shape. Light travels in straight line only in respect with the curved plane. So, altogether light travels in curved path and not in straight path. This explanation is more relevant and as it deals with curved plane, geodesics may be used here.

REFERENCES

[1]. Tietze, H., 1965. Famous problems of Mathematics. Graylock press 1, pp. 26-27.

[2]. Weinstein, E., 1974. CRC Concise Encyclopedia of Mathematics. CRC press, pp. 65.

[3]. Bennis, C.,Vezien, J., Iglesias, J., 1991. Piecewise Surface Flattening for Non-Distorted Texture Mapping. Computer Graphics, 25, 237-247.

[4]. Balasubramanian, M., Polimeni, J., 2009. Exact Geodesics and Shortest Paths on Polyhedral Surfaces. Pattern Analysis and Machine Intelligence, 1006-16.

[5]. Kiryati, N., Szekely, G., 1993. Estimating Shortest Paths and Minimal Distances on Digitized Three-Dimensional Surfaces Pattern Recognition. Pattern Recognition, 26, 1623-1637.

[6]. Zigelman, G., Kimmel,R., Kiryati,N., 2002. Computational Surface Flattening: A Voxel-Based Approach. Transactions on Pattern Analysis and Machine Intelligence, issue 04, 433-441.

[7]. Borg, I., Groenen, P., 1997. Modern Multidimensional Scaling: Theory and Applications. 14, issue 01, 3-36.

[8]. Lee, S., Han, J., Lee, H., 2006. Straightest Paths on Meshes By Cutting Planes. LNCS, 4077, 609-615.

[9]. Zigelman, G., Kimmel, R., Kiryati, N., 2002. Texture Mapping using Surface Flattening via Multi-Dimensional Scaling. Visualization and Computer Graphics (TVCG), 8, issue 02, 198-207.

[10]. Dale, A., Fischl, B., Sereno, M., 1999. Cortical Surface-Based Analysis: 2. Cortical Surface Based Analysis. NeuroImage, 9, 195-207.

[11]. Carmo, M., 1976. Differential Geometry of Curves and Surfaces. Prentice-Hall.

[12]. Drury, H., Van Essen, D., Anderson, C., Lee, C., Coogan, T., Lewis, J., 1996. Computerized Mappings of the Cerebral Cortex: A Multiresolution Flattening Method 24 and a Surface-Based Coordinate System. Journal of Cognitive Neuroscience, 8, 1-28.

[13]. Jonas, A., Kiryati, N., 1997. Digital Representation Schemes for 3-D Curves. Pattern Recognition, 30, 1803-1816.

[14]. Jonas, A., Kiryati, N., 1998. Length Estimation in 3-D using Cube Quantization. Journal of Mathematical Imaging and Vision, 8, 215-238.

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